THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050A Mathematical Analysis I (Fall 2021)
Suggested Solution of Test 2

If you find any errors or typos, please email me at
yzwang@math.cuhk.edu.hk

1. (25 points) Let $x_1 = 1$ and $x_{n+1} = 3 - x_n^{-1}$ for all $n \in \mathbb{N}$.

   (a) Show that $x_n \geq 1$ for all $n \in \mathbb{N}$;

   (b) Show that x_n is monotonic non-decreasing;

   (c) Show that $\{x_n\}_{n=1}^{\infty}$ is convergent;

   (d) Find the limit of $\lim_{n\to+\infty} x_n$.

Solution:

(a) We prove by induction that $x_n \geq 2$ for all $n \in \mathbb{N}$.

When $n = 1$, $x_1 = 1 \geq 1$.

Suppose $x_k \geq 1$ for some $k \in \mathbb{N}$. Then $x_{k+1} = 3 - x_k^{-1} \geq 3 - 1 \geq 1$. Hence $x_n \geq 1$ for all $n \in \mathbb{N}$.

(b) We prove by induction that $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$.

When $n = 1$, $x_2 = 2 \geq x_1$.

Suppose $x_{k+1} \geq x_k$ for some $k \in \mathbb{N}$. Since $x_n \geq 1$ for all $n \in \mathbb{N}$, we have that $x_{k+2} = 3 - x_{k+1}^{-1} \geq 3 - x_k^{-1} = x_{k+1}$. Hence $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$.

(c) Since $x_n \geq 1 > 0$ for all $n \in \mathbb{N}$, we have that $x_{n+1} = 3 - x_n^{-1} < 3$ for all $n \in \mathbb{N}$. Therefore $\{x_n\}$ is monotone non-decreasing and bounded from above. It follows that $\{x_n\}$ is convergent.

(d) Let $\lim_{n\to\infty} x_n = L \geq 1$. By taking limits on both sides of the equation $x_{n+1} = 3 - x_n^{-1}$, since $L > 0$, we have that $L = 3 - \frac{1}{L}$. Then $L = \frac{3-\sqrt{5}}{2}$ or $L = \frac{3+\sqrt{5}}{2}$. Since $L \geq 1$, we reject the solution $L = \frac{3-\sqrt{5}}{2}$. Hence,

$$\lim_{n\to+\infty} x_n = \frac{3+\sqrt{5}}{2}.$$

2. (25 points) (a) State without proof the Bolzano-Weierstrass Theorem;
   (b) State the definition of Cauchy sequence;
   (c) Prove that a sequence $\{x_n\}_{n=1}^{\infty}$ of real number is convergent if and only if it is a Cauchy sequence.

Solution:

(a) Suppose $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence, then there is a convergent subsequence.

(b) A sequence $\{x_n\}_{n=1}^{\infty}$ is said to be Cauchy if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n > N$,

$$|x_m - x_n| < \epsilon.$$

(c) Suppose $\{x_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} x_n = L$. Then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$|x_n - L| < \frac{\epsilon}{2}.$$

Hence for all $m, n > N$,

$$|x_m - x_n| \leq |x_m - L| + |x_n - L| < \epsilon.$$

Suppose $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Then there exists $N' \in \mathbb{N}$ such that for all $m, n > N'$,

$$|x_m - x_n| < 1.$$

It follows that $x_n \leq \max\{x_1, x_2, \ldots, x_{N'-1}, x_{N'} + 1\}$ for all $n \in \mathbb{N}$. Hence $\{x_n\}_{n=1}^{\infty}$ is bounded. By the Bolzano-Weierstrass Theorem, we can find a subsequence $\{x_{m_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ converging to some $L \in \mathbb{R}$. Now we fix an $\epsilon > 0$. Then there exists $N_1 \in \mathbb{N}$ such that for all $k > N_1$,

$$|x_{m_k} - L| < \frac{\epsilon}{2}.$$

Using the Cauchy assumption, for the same $\epsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that for all $m, n > N_2$,

$$|x_m - x_n| < \frac{\epsilon}{2}.$$

Let $N = \max\{m_{N_1}, N_2\}$. We can find some $k' > N_1$ such that $m_{k'} > N \geq N_2$. Then for all $n > N$,

$$|x_n - L| \leq \left|x_n - x_{m_{k'}}\right| + \left|x_{m_{k'}} - L\right| < \epsilon.$$

3. (30 points) Using ε-δ terminology or the sequential criterion to show that

(a) $\lim_{x\to 0} \dfrac{x^2+2}{x^2-2} = -1$;

(b) $\lim_{x\to+\infty} \sin(x)$ diverges.

Solution:

(a) Let $f(x) = \frac{x^2+2}{x^2-2}$. Note that for $|x| < 1$, $|x^2-2| > |1-2| = 1$ and $\frac{1}{|x^2-2|} \leq 1$. For any $\epsilon > 0$, we choose $\delta = \min\left\{\frac{\epsilon}{2}, 1\right\}$. Then for $|x - 0| < \delta$, we have that

$$|f(x)+1| = \left|\frac{2x^2}{x^2-2}\right| \leq 2x^2 < 2|x| < \epsilon.$$

Therefore, $\lim_{x\to 0} f(x) = -1$.

(b) Let $f(x) = \sin(x)$, $a_n = 2n\pi$ and $b_n = 2n\pi + \frac{\pi}{2}$. Note that for all $n \in \mathbb{N}$,

$$f(a_n) = \sin(2n\pi) = 0$$

and

$$f(b_n) = \sin(2n\pi + \frac{\pi}{2}) = 1.$$

Then $\lim_{n\to\infty} f(a_n) = 0 \neq 1 = \lim_{n\to\infty} f(b_n)$. Since $b_n > a_n > n$ for all $n \in \mathbb{N}$, we have that both a_n and b_n go to ∞ as $n \to \infty$. Therefore, $\lim_{x\to\infty} \sin(x)$ does not exist.

4. (20 points) Suppose $f : (0, +\infty) \to \mathbb{R}$ is a function given by

$$f(x) = \begin{cases} \dfrac{1}{m^2}, & \text{if } x = \dfrac{m}{n}, \ \gcd(m,n) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Find the set of $c \in \mathbb{R}^+$ where f is continuous at c. Give your reasoning.

Solution: We first show that f is discontinuous on $\mathbb{Q}^+ = \mathbb{R}^+ \cap \mathbb{Q}$.

For any $q \in \mathbb{Q}^+$, we can find a sequence of positive irrational numbers $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x_n = q$. Then $f(x_n) = 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} f(x_n) = 0$. Since $f(q) = \frac{1}{m^2} > 0$ for some $m \in \mathbb{N}$, we have that $f(x)$ is not continuous at $x = q$.

Then we prove that f is continuous on $C = \mathbb{R}^+ - \mathbb{Q}$. Let $c \in C$ and $\epsilon > 0$. We define a set

$$S_{\epsilon,c} := \left\{ x \in (\frac{c}{2}, \frac{3c}{2}) : x = \frac{m}{n}, \ \gcd(m,n) = 1 \text{ and } \frac{1}{m^2} \geq \epsilon \right\}.$$

We claim that $|S_{\epsilon,c}|$ is finite for any $\epsilon > 0$ and $c \in C$. Suppose our claim is true. Since $S_{\epsilon,c} \subset \mathbb{Q}$ and $c \in \mathbb{R}^+ - \mathbb{Q}$, there exists $\delta > 0$ such that

$$(c - \delta, c + \delta) \cap S_{\epsilon,c} = \emptyset.$$

It follows that for any $x \in (c - \delta, c + \delta)$,

$$f(x) = \begin{cases} \frac{1}{m^2} < \epsilon & \text{if } x = \frac{m}{n}, \ \gcd(m,n) = 1; \\ 0 < \epsilon & \text{otherwise.} \end{cases}$$

Hence $|f(x) - 0| = f(x) < \epsilon$ for any $|x - c| < \delta$. Therefore $\lim_{x\to c} f(x) = 0 = f(c)$, $f(x)$ is continuous at $x = c$.

Now it remains to prove the claim.

For any $x \in S_{\epsilon,c}$, there eixsts a unique pair of natural numbers m, n such that $x = \frac{m}{n}$ and $\gcd(m,n) = 1$. By the definition of S, the numbers m, n satisfy that

$$\frac{c}{2} < \frac{m}{n} < \frac{3c}{2} \text{ and } \frac{1}{m^2} \geq \epsilon.$$

Rewriting the inequalities yields that

$$\begin{cases} \frac{2m}{3c} < n < \frac{2m}{c} \\ m \leq \frac{1}{\sqrt{\epsilon}}. \end{cases}$$

It follows that both m, n are bounded. Therefore $|S_{\epsilon,c}|$ is finite.